Homework Assignment 3

ENM 502 – Numerical Methods

Due Thursday 3/27/2014

A. General Project Statement

Consider the following **non-linear** boundary-value problem defined on the unit-square domain 

 (1)

 on all boundaries (),

where  and  are (known) parameters.

You are to first discretize eq. (1) and boundary conditions using centered finite difference approximations and thereby generate a set of coupled, non-linear algebraic equations of the form . This system of equations should be solved using Newton’s Method for given values of the parameters  and . The overall goals of the project are as follows:

(a) Track all (there are several) non-trivial solutions for  using both analytic and arc-length continuation methods. You should implement analytic continuation first. For arc-length continuation, see Supplementary Notes below.

(b) Repeat (a) for the case where . It is suggested that you use arc-length continuation for this part.

B. Numerical Method Considerations

(i) Use the intrinsic MATLAB functions (“lu” and “\”) to numerically solve any linear system of equations that arise. Implement the code so that analytic continuation can be used in either of the two parameters. The supplementary document attached to this assignment provides step-by-step details for implementing arc-length continuation.

(ii) Use a 30x30 uniform finite difference grid for all calculations.

(iii) For the  case, first use the solutions to the linearized problem to jump onto the various solution branches. Then use analytic/arc-length continuation to trace out the family of solutions as a function of .

(iv) For the  case, it is best to use the  solutions and analytic continuation in  to help you jump on to the various branches.

C. Additional Notes

(i) Your major result should be summarized on a plot of  versus  for all solution branches in the interval  for each value of . Note that while the 2-norm is always positive, it is useful for clarity to assign “negative” and “positive” sub-branches for a given branch of the solution. For example, “hill-type” solutions could be positive and “bowl-like” solutions. Whatever convention you decide to use, you should make sure that you clearly define it in your report.

(ii) Plot representative contour maps of the solutions found along the different solution branches so that it is clear how the solution changes along each branch. This should be described in your report. It is suggested that you use contour plots for all 2-dimensional plots with equi-value contour lines to represent the solution – do not use filled color maps unless you have access to a color printer. Make sure that your axes are labeled and that the contour lines are defined by a legend or are individually labeled. Refer to the Style Guide to help you write your report.

(iii) A discussion should be provided on how the forcing function in eq. (1) changes the overall solution structure. In this context, SKETCH (no need for additional calculations) the solution structure you expect for .

(iv) Hand in printouts of your MATLAB code along with the rest of the report. It is in your best interest to provide as much documentation with your codes as possible.

D. Logistics

(i) First, implement the Newton method without regard for continuation. This should consist of an M-file that contains the Newton loop in which calls are made to routines that build the Jacobian and Residual at a given solution approximation. Test this code using any non-linear equation system but make sure it’s working before proceeding.

(ii) Call the Newton routine from another M-file that contains the continuation code. Begin with analytic continuation. The continuation code should consist of an outer loop that steps in the value of the parameter . At each value, you need to used a previously stored solution to generate a good guess for the solution at the next value. Each solution branch will require a separate sequence of runs.

(iii) Using essentially the same approach, develop a separate code to perform arc-length continuation.

SUPPLEMENTARY NOTES for ARC-LENGTH CONTINUATION

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Use the following approach to begin your arc-length continuation (ALC) solution tracking:

1. Start with an initial guess for 

2. Using Newton’s Method get the converged solution at  at  (you select the initial value of ).

3. Use one step of analytic continuation (AC) to obtain the converged solution at  (you select the step size in ).

4. Once you have the solution at, you can use ALC to obtain the solution as a function of as described below. Note that you will need the two previous values to perform ALC as shown in the figure below and described in more detail next.







**Figure 1**: Initiating arc-length continuation using the solution at **two** previous points.

# Arc-Length Continuation: ALC

The initial guess forusing ALC is given by:

 (1)

where *S* is any general arc-length along the curve where we want to find the solution and *SR* is the reference arc length, such that *S-SR* is small, and to a good approximation



Now consider the situation where, after implementing the first four steps on the previous page, we want to find the solution (i.e., at point “2” in the figure). Here,



But before we can use eq. (1) as intelligent initial guesses for  we need to evaluate

,

which, for point “2”, translate to

.

Make sure that you follow the subscript notation – the index SR is not fixed, but refers to the solution at the current point from which we are about to move forward in arc-length.

The preceding quantities can be evaluated from the following equation obtained by expanding the augmented residual, , in a Taylor Series at point *S* about some reference point *SR*:

 (2)

where

 (3)

and





where **** is the arc-length at the point previous/next to the current point (*S*). The augmented residual derivative w.r.t *S* is then given explicitly by (make sure that you understand this!!).

 (4)

Thus,

 (5)

Now to find the quantities , i.e. at  and  we can write

 (6)

which is just eq. (5) rewritten at a specific value of arc-length.

Using above expressions for  in eqs. (3) and (4), we can now compute

.

The latter expressions can then be substituted back into eq. (1), to obtain the initial guess for at point “2”. This completes a full cycle of the arc-length continuation scheme and further stepping can be achieved by repeating the above sequence of steps. Note once again that the computation of the initial guess at point “2” we needed to know the solution at previous two points, i.e. “0” and “1”. Finally, for reference, the equations for the Full Newton Method are provided below.

**Full Newton’s Method with an Augmented Residual/Jacobian:**

Once we obtain the initial guess at point we need to iterate using the Newton’s Method till both are below the suggested tolerance values.

For the Augmented system the equation for Full Newton’s Method looks like:

